Applications of a Quadratic Extended Interior Penalty Function for Structural Optimization

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A quadratic extended interior penalty function formulation especially well-suited for second-order unconstrained optimization procedures is presented. Analytical derivatives of constraints and an approximate analysis technique are used. Minimum-mass design results are presented which indicate that the combination of these procedures can help make mathematical programming a useful optimization tool for large-order structural design problems with a large number of design variables and multiple constraints. Examples include statically loaded high- and low-aspect-ratio wings simultaneously subjected to stress, displacement, minimum gage and, in some cases, maximum strain constraints.

Introduction

EFFICIENT and dependable designs of increasing complex new aerospace vehicles demand improved structural optimization procedures. Mathematical programming techniques are one type of procedure used since the early 1960's[1] for determining minimum-mass designs of structures or structural components subjected to various design constraints such as stress, displacement, and minimum gage. Because of its relative simplicity, the sequence of unconstrained minimizations technique, SUMT,[2] is a popular mathematical programming technique. By introducing the design constraints as penalty functions, the SUMT method transforms a constrained minimization problem into an unconstrained problem. In the standard SUMT procedure, the design process is initiated in and must remain in the feasible, or acceptable, design space. Each unconstrained minimization is usually performed (see, for example, Refs. 3-6) by a quasi-Newton algorithm such as the Davidon-Fletcher-Powell method.[7]

The main disadvantage of quasi-Newton algorithms is that the number of structural analyses required for the optimization procedure (one measure of computational efficiency) is a linear function of the number of design variables. These algorithms, therefore, are not suitable for use with a large number of design variables. It was recently shown,[8] however, that Newton's method applied with an interior penalty function formulation can be used to overcome this disadvantage because it is possible to obtain a simple approximation to the second derivatives of the penalty function necessary for Newton's method. This method requires a small number of structural analyses that is independent of the number of design variables and can, therefore, be useful for solving structural design problems involving a large number of design variables.

The use of an efficient optimization algorithm is not sufficient by itself to make the automated resizing of a complex structural system a tractable problem. Approximate analysis techniques[9] are often used to further increase computational efficiency. As a result of the use of approximate methods, it is possible for a design determined by the optimization procedure to fall outside of the feasible design domain. Therefore, standard interior penalty function methods may not always be adequate by themselves. Problems associated with an infeasible design point can be overcome by changing to an exterior penalty function formulation, where necessary, or by the use of an extended interior penalty function formulation such as the linear extended interior penalty function presented in Ref. 10.

The present paper describes a quadratic extended interior penalty function which preserves the continuity of the second derivatives throughout the design space and is, therefore, especially well-suited for use in conjunction with Newton's method. The behavior of the quadratic extended penalty function is investigated and the permissible range of the parameters that define this extended penalty function are discussed.

The optimization procedure is used to obtain minimum-mass designs of structures subjected to stress, strain, displacement, and minimum gage constraints. Derivatives of the constraints with respect to design variables are obtained analytically. These derivatives are used in the optimization procedure to determine search directions, and to obtain approximate expressions for the constraint functions along the one-dimensional search paths.

Newton's method coupled with analytical derivatives, a quadratic extended interior penalty function, and approximation techniques reduce computational times, and increase the order of the structural design problem that can be solved. Several example wing structures are used to demonstrate the optimization procedure.

Analytical Procedures

Interior Penalty function

The constrained optimization problem considered herein is to find a vector of design variables v^* that minimizes the mass $m(v)$ of a structure subject to the constraints

$$g_i(v) \geq 0; \quad i=1,\ldots, n \tag{1}$$

It is assumed that at least one of the constraints of Eq. (1) is critical at the minimum v^*, that is, $g_i(v^*)=0$ for some i. This

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constrained optimization problem may be transformed into a series of unconstrained minimization problems by introducing a penalty function associated with the constraints, and the transformed problem can be solved by the sequence of unconstrained minimizations technique. The resulting transformed problem is to find the minimum of a function $P(r)$ as r goes to zero where

$$P(r) = m(v) + r\sum_{i=1}^{n} f_i(v) \tag{2}$$

and $f_i(v)$ is defined here as

$$f_i(v) = 1/g_i(v) \quad \text{if} \quad g_i(v) > 0 \tag{3}$$

The term $rf_i(v)$ represents the penalty associated with the ith constraint, and is an interior penalty function in the sense that it is defined only if v is inside the feasible design domain. With v^r denoting the point in design space where $P(r)$ attains its minimum value for a given value of r, it may be shown (Ref. 2, pp. 47-48) that as r goes to zero

$$\text{min.} \quad P(r) \to m(v^*)$$

$$v^r \to v^*$$

The behavior of the $g_i(v^r)$, as r goes to zero can be investigated by defining a function

$$g = \left(\sum_{i=1}^{n} 1/g_i\right)^{-1} \tag{4}$$

To determine this behavior, the following assumptions can be made for a vector v which has the components $v_1, \ldots, v_N$

1) $m(v)$ and $g_i(v)$, $i = 1, \ldots, n$, are continuously differentiable functions

2) two positive constants, k_1, and r_0, can be found at $v = v^r$ for at least one j such that‡

$$\infty > k_1 > \left| \frac{\partial m}{\partial v_j} \Big/ \frac{\partial g}{\partial v_j} \right| > 0 \quad \text{for all} \quad r < r_0$$

By using Eqs. (3) and (4), it is possible to write Eq. (2) as

$$P(r) = m + r/g \tag{5}$$

Since P is a minimum at v^r

$$\frac{\partial P}{\partial v_j} = \frac{\partial m}{\partial v_j} - \frac{r}{g^2}\frac{\partial g}{\partial v_j} = 0; \; j = 1, \ldots, N \tag{6}$$

Solving for g gives

$$g(v^r) = r^{1/2}\left(\frac{\partial g}{\partial v_j} \Big/ \frac{\partial m}{\partial v_j}\right)^{1/2} \tag{7}$$

Equations (4) and (7) give

$$1/g(v^r) = \sum_{i=1}^{n} 1/g_i(v^r) = \left(\frac{\partial m}{\partial v_j} \Big/ \frac{\partial g}{\partial v_j}\right)^{-1/2} r^{-1/2} \leq k_1 r^{-1/2} \tag{8}$$

Equation (8) implies that each g_i in the summation satisfies

$$g_i \geq k_2 r^{1/2} \tag{9}$$

where k_2 is some positive constant, and consequently $g_i(v^r)$, $i = 1, \ldots, n$, does not go to zero faster than $r^{1/2}$.

‡If this condition is not satisfied at $v = v^r$, then it is possible that the mass can be changed without affecting any of the constraints, a situation not likely to occur for a practical structural design problem.

Extended Interior Penalty Function

The interior penalty function formulation of Eqs. (2) and (3) has the disadvantage that the initial design point must be feasible and that the design must remain in the feasible design domain throughout the optimization process. When approximate analysis methods are used, the second requirement is sometimes difficult to satisfy because the approximate nature of the analyses can cause the constraints to be violated. Extending the penalty function formulation into the infeasible design domain provides an efficient mechanism to recover from violations brought about by approximate analyses and also permits an infeasible initial design.§

A linear extended interior penalty function proposed in Ref. 10 permits the use of design points that are outside of the feasible domain. For this linear extended penalty function, the quantities f_i in Eq. (2) are

$$f_i = \begin{cases} 1/g_i & \text{if } g_i \geq g_0 \\ 1/g_0[2 - g_i/g_0] & \text{if } g_i \leq g_0 \end{cases} \tag{10}$$

The transition point g_0 between the two constraint functions in Eq. (10) varies as

$$g_0 = Cr \tag{11}$$

where C is a constant

Because the f_i of Eq. (10) have discontinuous second derivatives at the transition point, the linear extension of the penalty function is not suitable for a second-order (e.g., Newton's method) optimization algorithm. This disadvantage can be overcome by introducing a quadratic extended interior penalty function that is continuous and has continuous first and second derivatives. The definition of the f_i in Eq. (2) for the quadratic extended penalty function is

$$f_i = \begin{cases} 1/g_i & \text{if } g_i \geq g_0 \\ 1/g_0[(g_i/g_0)^2 - 3(g_i/g_0) + 3] & \text{if } g_i \leq g_0 \end{cases} \tag{12}$$

where g_i must also have continuous first and second derivatives. The quadratic and linear extensions to the penalty function are illustrated in Fig. 1 along with a conventional interior penalty function for comparison.

To complete the definition of the quadratic extended penalty function, a relation that defines the transition point g_0 between the two constraint functions in Eq. (12) is required. As r goes to zero, the following two conditions should be satisfied.

$$rf_i \to 0 \quad \text{for any} \quad g_i > 0$$

which represents a vanishing contribution in the feasible domain

$$rf_i \to \infty \quad \text{for any} \quad g_i < 0$$

which represents an increasing penalty for constraint violation. It can be easily verified that the first condition is satisfied because $f_i \leq 1/g_i$ and r/g_i goes to zero as r goes to zero for any given value of g_i. The second condition is equivalent to the requirement that

$$r/g_0{}^3 \to \infty \quad \text{as} \quad r \to 0 \tag{13a}$$

or

$$g_0/r^{1/3} \to 0 \quad \text{as} \quad r \to 0 \tag{13b}$$

§There are, of course, numerous recovery schemes that can bring a design from infeasible design space back to feasible design space. But it is believed that these recovery schemes are not as efficient as the extended penalty function approach.

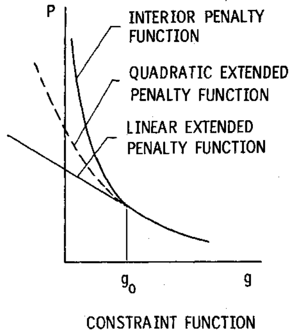

Fig. 1 Penalty functions.

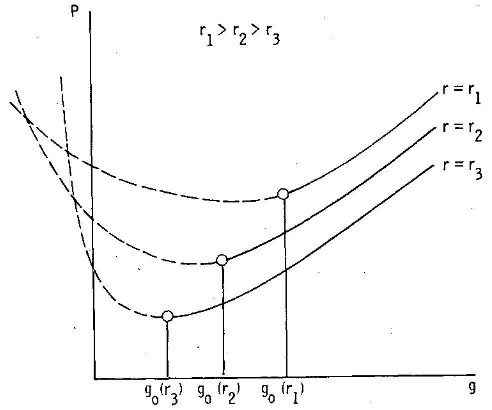

Fig. 2 Quadratic extended penalty function for different values of r.

Equation (13b) defines one limit on the behavior of g_0. Another limit can be determined by requiring the design point to move in the range where f_i is defined quadratically most of the time rather than as $1/g_i$. This requirement is desirable because the quadratic definition of f_i may be expected to be better behaved than the $1/g_i$ form of the function. As can be seen from Eq. (9), g_i does not go to zero faster than $r^{1/2}$ for a conventional penalty function formulation and, therefore, if g_0 goes to zero faster than $r^{1/2}$, the minimum of $P(r)$ will drift away from the quadratic range as r goes to zero. It is assumed, therefore, that the transition point g_0 for the quadratic extension to the penalty function can be expressed as

$$g_0 = Cr^p; \quad 1/2 \geq p > 1/3 \tag{14}$$

where C is a constant. In summary, the exponent 1/3 in Eq. (14) guarantees that the penalty for violating the constraints increases as r goes to zero while the exponent 1/2 is needed to help keep the minimum point v^r in the quadratic range of the penalty function. A representation of the quadratic extended penalty function for different values of r is shown in Fig. 2.

Newton's Method with Approximate Second Derivatives

To apply Newton's method with the SUMT procedure, the point v^r that minimizes the function $P(r)$ for a given value of r is found by using an iterative procedure. If v^i is an initial guess for v^r, a better approximation v^{i+1} is found from

$$v^{i+1} = v^i - \alpha H^{-1} \nabla P \tag{15}$$

where ∇P is the gradient of P, H is the matrix of second derivatives of P given by

$$H_{jk} = \frac{\partial^2 P}{\partial v_j \partial v_k} \tag{16}$$

and α is the step size from v^i to v^{i+1} found by means of a one-dimensional search in the direction of $H^{-1} \ \nabla P$. As shown in Ref. 8, the second derivative matrix H may be approximated by using only the first derivatives of the constraint functions. Then, following Ref. 8, the second derivative of Eq. (2) with respect to the design variables can be written as

$$\frac{\partial^2 P}{\partial v_j \partial v_k} = \frac{\partial^2 m}{\partial v_j \partial v_k} + r \sum_{i=1}^{n} \frac{\partial^2 f_i}{\partial v_j \partial v_k} \tag{17}$$

and from Eq. (12), for the quadratic extended penalty function,

$$\frac{\partial^2 f_i}{\partial v_j \partial v_k} = g_i^{-3} \left[2 \frac{\partial g_i}{\partial v_j} \frac{\partial g_i}{\partial v_k} - g_i \frac{\partial^2 g_i}{\partial v_j \partial v_k} \right] \quad \text{if } g_i \geq g_0 \tag{18a}$$

$$\frac{\partial^2 f_i}{\partial v_j \partial v_k} = g_0^{-3} \left[2 \frac{\partial g_i}{\partial v_j} \frac{\partial g_i}{\partial v_k} + g_0 \left(2\frac{g_i}{g_0} - 3 \right) \frac{\partial^2 g_i}{\partial v_j \partial v_k} \right] \quad \text{if } g_i \leq g_0 \tag{18b}$$

Because of the factors g_i^{-3} and g_0^{-3} in Eq. (18), the main contribution to the penalty function second derivatives is from the constraints which are nearly critical¶ (i.e., g_i very small). For these constraints, the following approximations for the second derivatives of the quadratic extended penalty function may be used

$$\frac{\partial^2 f_i}{\partial v_j \partial v_k} = 2 g_i^{-3} \frac{\partial g_i}{\partial v_j} \frac{\partial g_i}{\partial v_k} \quad \text{if } g_i \geq g_0 \tag{19a}$$

$$\frac{\partial^2 f_i}{\partial v_j \partial v_k} = 2 g_0^{-3} \frac{\partial g_i}{\partial v_j} \frac{\partial g_i}{\partial v_k} \quad \text{if } g_i \leq g_0 \tag{19b}$$

This approximation requires only the first derivatives of the constraints; therefore, the computational effort needed to generate the direction cosines for a one-dimensional search using Newton's method is similar to that required by quasi-Newton methods which use only first derivatives of $P(r)$. In addition, whereas the number of one-dimensional searches required by Newton's method is small and independent of the number of design variables, the number of one-dimensional searches required by a quasi-Newton method is proportional to the number of design variables. For these reasons Newton's method provides increased efficiency for problems having a large number of design variables.

Analytical Derivatives

For statics problems, the first derivatives of most of the constraint functions needed for Eq. (19) require the derivatives of stresses and displacements with respect to the design variables. Because these derivatives can be computed analytically, the large number of analyses associated with finite-difference derivatives calculation can be avoided. For a problem with N design variables, finite-difference derivative calculations require $N+1$ analyses to generate a direction for each one-dimensional search. By using analytical derivatives only a small number of analyses are required during the design process. For the examples presented in this paper one-*complete* analysis, which includes all derivative calculations, is performed for each one-dimensional search. Since the time required for these derivative calculations is smaller than the time required for an analysis, analyatical derivatives substantially reduce the computational times required by the optimization procedure.

Since the constraints that are treated in the example problems involve displacements and stresses, the derivatives of the displacement vector and the stresses are needed for the computation of the derivatives of $P(r)$. The derivatives of the

¶It is assumed here that g_0 is small. Note that even if g_0 were not very small for the first value of r it will become smaller as $r \rightarrow 0$, see Eq. (14).

displacement vector u with respect to a design variable v_i can be found from

$$Ku = L \tag{20}$$

where K is the stiffness matrix and L is the applied load vector. Differentiating Eq. (20) leads to

$$\frac{\partial u}{\partial v_i} = -K^{-1}\left[\frac{\partial K}{\partial v_i}u - \frac{\partial L}{\partial v_i}\right] \tag{21}$$

For problems where L is independent of the design variables, Eq. (21) reduces to

$$\frac{\partial u}{\partial v_i} = -K^{-1}\frac{\partial K}{\partial v_i}u \tag{22}$$

The derivative of the stresses σ with respect to a design variable can be found from

$$\sigma = Su \tag{23}$$

where S is independent of the design variables. Differentiating Eq. (23) gives

$$\frac{\partial \sigma}{\partial v_i} = S\frac{\partial u}{\partial v_i} \tag{24}$$

where $\partial u/\partial v_i$ is given by Eq. (22).

Temporarily Inactivated Design Variables

The number of design variables can be reduced during portions of the optimization process by eliminating those design variables that are governed only by a minimum gage constraint. This is done by temporarily inactivating a design variable when it becomes equal to its minimum gage. Periodic checks are made to determine whether the search procedure requires that this design variable be increased above its minimum gage value and, consequently, reactivated. Between such checks there is no need to calculate derivatives with respect to this design variable. An extended penalty function facilitates the use of inactivated design variables because it allows the design variables to achieve minimum gage values. If a conventional interior penalty function were used, design variables controlled by minimum gage constraints would be slightly larger than their minimum gage values. As a result, it would be difficult to identify these design variables during the early stages of the optimization process for the purpose of inactivating them.

Approximate Analyses During One-Dimensional Search

A further reduction in the number of analyses required to find a minimum-mass design can be obtained by using approximate analysis techniques during the one-dimensional search portion of the optimization process. By making approximations to the constraints, a new analysis at each design point along the one-dimensional search path can be avoided. Since stresses, strains, and displacements are inversely proportional to the thicknesses for statically determinate structures, a logical approximation** for the associated constraints is

$$g_i(v) = C_{0i} + \sum_{k=1}^{N}\frac{\alpha_{ik}}{t_k(v)} \tag{25}$$

Differentiating Eq. (25) gives an approximate expression for the derivative of $g_i(v)$

$$\frac{\partial g_i(v)}{\partial v_k} = -\frac{\alpha_{ik}}{[t_k(v)]^2} \tag{26}$$

In Eqs. (25) and (26) C_{0i} and α_{ik} are constants to be determined. The design variable v_k in the optimization procedure is the variable thickness above minimum gage so that the total thickness t_k is given by

$$t_k = t_{\min_k} + v_k \tag{27}$$

where $t_{\min_k}$ is the minimum gage for this design variable.

The constants C_{0i} and α_{ik} can be determined by matching the g_i and $\partial g_i/\partial v_k$ at the initial point v_0 of each one-dimensional search. Accurate values of g_i and $\partial g_i/\partial v_k$ are known at v_0 because, in the approach used herein, a complete analysis (which includes the calculation of analytical derivatives) is carried out at that point. On that basis, Eqs. (25) and (26) give

$$\alpha_{ik} = -[t_k(v)]^2\frac{\partial g_i(v_0)}{\partial v_k} \tag{28}$$

and

$$C_{0i} = g_i(v_0) - \sum_{k=1}^{N}\frac{\alpha_{ik}}{t_k(v_0)} \tag{29}$$

Substituting Eqs. (28) and (29) into Eq. (25) gives the desired approximation

$$g_i(v) = g_i(v_0) + \sum_{k=1}^{N} t_k(v_0)\frac{\partial g_i(v_0)}{\partial v_k}\left[1 - \frac{t_k(v_0)}{t_k(v)}\right] \tag{30}$$

Equation (30) is used only during a one-dimensional search. Direction cosines for a one-dimensional search are obtained using an accurate analysis.

Because the approach described in the preceding approximate technique, it can lead to violations of the constraints. The quadratic extended interior penalty function provides an efficient and convenient recovery mechanism for such constraint violations.

An additional reduction in computational time can be achieved by considering for each one-dimensional search only those constraints which are critical or near critical at the beginning of the search. At a given point in design space a constraint is considered to be critical or near critical if the corresponding g_i is close to the value of the minimum g_i. This procedure (similar to the truncated posture table of Ref. 9) also results, occasionally, in violation of constraints and benefits from the use of the extended penalty function formulation.

Computer Program

The computer program used for performing the calculations for the examples given in this paper is a modification of the WIDOWAC program.[11] In the program, a wing structure is modeled by a finite-element representation that includes rod elements, constant-strain membrane elements for cover panels, and shear web elements for ribs and spars. The program is limited to symmetric airfoils. Design variables define thicknesses or areas of structural elements, and it is possible to assign the same design variable to more than one element. It is also possible to divide part or all of the wing skin into segments containing one or more membrane cover-panel elements with a linear variation of thickness within each element of the segment. Thus a single design variable may control the thickness of a single element, a group of elements, or a vertex of a segment.

One-dimensional searches are conducted by using a parabolic approximation to the penalty function along a search direction. This approximation is based on the three best design points previously determined during a given one-dimensional search. A search is terminated when any of the following three criteria are satisfied: 1) 20 steps have been

**The idea for this approximation is based on the work with inverse design variables in Ref. 9.

taken in the search direction; 2) the expected improvement in the object function $P(r)$ based on the parabolic approximation is less than 0.05%; or 3) the step size is reduced to less than 0.05 times the first step size.

For the example problems described in the following, Eq. (14) was used based on an initial value of g_0 between 0.1 and 0.3 and a value of p equal to 0.5. An initial value of r was chosen that made the penalty function approximately equal to the mass. After each unconstrained minimization r was reduced by a factor that was between 10 and 30. The optimization procedure was terminated when the penalty function was reduced to less than 2% of the mass.

Description of Wing Models

In this study two models of a low-aspect-ratio wing and one model of a high-aspect-ratio wing are investigated. Since the computer program used in this study was developed for symmetric airfoils, only the upper half of the wing surfaces needs to be modeled. The constraints applied to the example problems presented herein are stress, strain, displacement, and minimum gage. Since finite-element models are used to represent the example structures, displacement constraints are applied at specified nodes of the finite-element model.

Low-Aspect-Ratio Wing

A clamped, titanium, clipped-delta wing with a sândwich core having infinite transverse shear stiffness is studied as an example of a low-aspect-ratio wing and is shown in Fig. 3. the wing is subjected to a uniform pressure loading and is required to satisfy a maximum allowable stress requirement. The normal displacements of 10 points on the outboard portion of the wing are required to be less than specified maximum values and the wing skin is subjected to a minimum gage constraint.

Basic Model:

A basic finite-element model represents the wing structure with 93 degrees of freedom. The basic model consists of 51 triangular membrane cover-panel elements, and, to simulate the rigid core, 49 very stiff constant-thickness shear web elements. The basic model is studied using seve, 20, 28, and 51 design variables. For the seven- and 20-design-variables studies, the cover-panel elements are grouped into six segments that are represented in Fig. 3 by the dashed lines. The design variables represent the change in the thicknesses at the segment vertices. For the seven-design-variable study, each design variable controls the thickness of several vertices; and for the 20-design-variable study, one design variable is assigned to each of 18 vertices and two design variables control the thicknesses of the four remaining vertices. For the 28-design-variable study, one design variable is assigned to the thickness of each of 17 elements in three segments and one to each of the vertices of the other three segments. For the 51-design-variable study, one design variable represents the change in thickness of each of the 51 cover-panel elements.

Refined Model:

A structural refinement of the basic model is also studied by using 82 design variables. The refined finite-element model consists of 82 triangular membrane cover-panel elements, 79 constant-thickness shear web elements, and has 138 degrees of freedom. One design variable represents the change in thickness of each of the 82 cover-panel elements.

High-Aspect-Ratio Wing

A graphite/epoxy composite wing consisting of ribs, spars, and cover panels is studied as an example of a high-aspect-ratio wing and is shown in Fig. 4. The wing is subjected to a uniform pressure loading and a concentrated engine load. Normal displacements of the wing are constrained to be zero along the fuselage boundary, and symmetry is imposed at the fuselage center line. Leading- and trailing-edge fairings are assumed to be inactive as structural members and a finite-element model represents the remaining wing box structure with 201 degrees of freedom. The cover panels of the structural box consist of composite material with 0°, 90°, and ±45° ply orientations and are represented by 192 membrane finite elements with 64 of these elements assigned to each of the three-ply orientations. The ribs and spars constst of ±45° material and are represented by 98 shear web elements. Constraints on the normal displacements of nine points in the outer half of the wing are imposed on the wing. The cover panels are subjected to minimum gage and maximum strain allowable requirements, and the ribs and spars are subjected to minimum gage and maximum strain allowable requirements, and the ribs and spars are subjected to minimum gage and maximum stress allowable requirements.

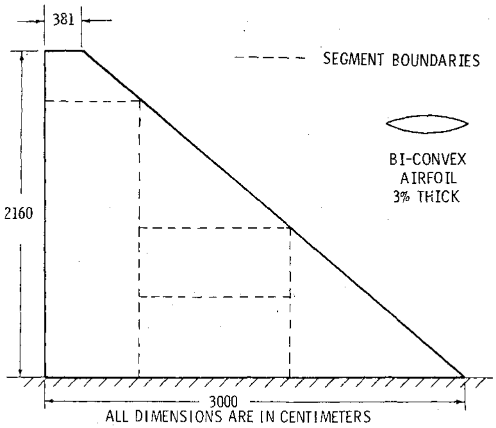

Fig. 3 Aerodynamic planform, airfoil section, and segments for low-aspect-ratio wing.

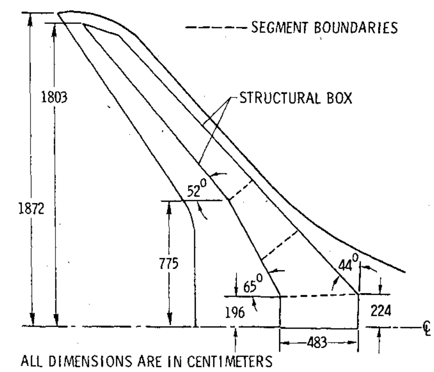

Fig. 4 Aerodynamic planform, structural box, and segments for high-aspect-ratio wing.

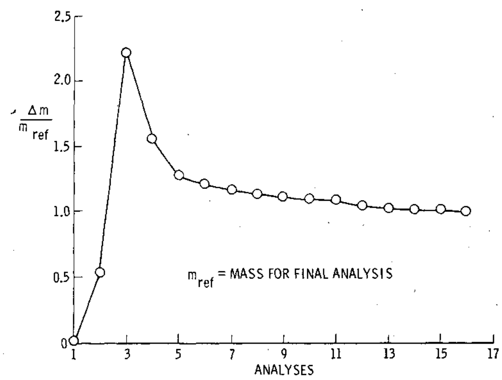

Fig. 5 Variable mass history using the quadratic extended interior penalty function for a seven-design-variable problem that initially violates constraints. Low-aspect-ratio wing.

Table 1 Design history of low-aspect-ratio wing started from minimum gage

Analysis	Δm, kg	g_i for most critical constraint[a]
1	0.	−2.0793
2	850.1	−0.6922
3	3542.3	.1829
4	2455.9	.0047
5	2035.3	− .0162
6	1915.3	− .0096
7	1851.8	− .0003
8	1798.9	.0108
9	1771.5	.0182
10	1751.5	.0245
11	1736.4	.0299
12	1644.8	.0094
13	1619.9	.0062
14	1611.9	.0047
15	1601.3	.0018
16	1594.3	.0005

[a]The g_i for the stresses, strains, and displacements are normalized by allowable values in such a way that a g_i equal to 0.1 means that a constraint is 10% below the allowable value as a g_i of −0.1 means that a constraint is 10% above the allowable value. Negative values of g_i indicate a constraint violation.

The high-aspect-ratio wing model is studied using 13, 25, 32, 50, 74, and 146 design variables. For these studies the cover-panel elements for each ply orientation are grouped into the four segments represented in Fig. 4 by the dashed lines. For the 13-design-variable study, one design variable is assigned to each ply orientation thickness in each segment and one is assigned to the thickness of the shear webs. For the 25-design-variable study, two design variables are assigned to each ply orientation thickness in each segment. For the 32- and 50-design-variable studies, eight and 26 design variables are assigned to the shear web thicknesses. For the 74- and 146-design-variable studies, 48 design variables are assigned to the cover panels, and 26 and 98 design variables are assigned to the shear webs, respectively.

Results and Discussion

One of the benefits of an extended penalty function is that the design can be started from the infeasible domain. For example, the low-aspect-ratio basic model was designed starting from a prescribed minimum gage configuration. Both the stress and displacement constraints were violated when all the thicknesses were set to minimum gage. The dependence of the variable mass Δm of the structure (not including the minimum gage mass) and the value of the g_i for the most critical constraint as a function of the number of complete analyses are given in Table 1. The variable mass Δm normalized by the variable mass for the final design m_{ref} is also shown in Fig. 5 as a function of the number of complete analyses. After three analyses the design reached the feasible domain with a variable mass of about 2.2 times that of the final design. In the other examples described in the following the designs were started in the feasible domain and any constraint violations that occurred were primarily due to the approximation or deletion of constraints during a one-dimensional search.

The wing models described in the previous section are designed using an increasing number of design variables. The total CPU times, the total number of unconstrained minimizations, the total number of analyses, and the final design total mass as a function of the number of design variables are given in Table 2 for the low-aspect-ratio wing and in Table 3 for the high-aspect-ratio wing. The dependence of the final total mass m on the number of design variables is illustrated in Fig. 6. In Fig. 6 the final total masses of the high-aspect-ratio wing designs are normalized by the mass m_{ref} of the 13-design-variable design, and the final total masses of the low-aspect-ratio wing designs are normalized by the mass m_{ref} of the seven-design-variable design. It is evident that increasing the number of design variables results in a considerable reduction in final mass.

The computer run times as a function of the number of design variables for the basic low-aspect-ratio wing model and the high-aspect-ratio wing model are shown in Fig. 7. The three main components of the computer time are 1) time for the solution for displacements and stresses which is independent of the number of design variables, 2) time for derivative calculations which is proportional to the number of design variables, and 3) calculation of search directions which requires the generation and decomposition of the second derivative matrix and involves terms that are quadratic and cubic in the number of design variables. The curves in Fig. 7 indicate that for up to 50 design variables the relation between run times and number of design varibles is close to linear which suggests that the first two components dominate. Above 50 design variables the third component starts to have some influence so that the curves are no longer linear, but, in these examples, the deviation from linearity is not very large.

The seven-design-variable case for the low-aspect-ratio wing model was also designed without displacement constraints and required 23 sec of CDC 6600 CPU time and 15

Table 2 Results for low-aspect-ratio wing studies

Number of design variables	CDC 6600 CPU time, sec	Total number of unconstrained minimizations	Total number of analyses	Final total mass, kg
7	24	5	16	3091.7
20	55	4	19	3026.8
28	77	4	23	2779.4
51	111	5	22	2756.6
82	393	5	24	2618.1

Table 3 Results of high-aspect ratio wing studies

Number of design variables	CDC 6600 CPU time sec	Total number of unconstrained minimizations	Tota number of analysis	Final total mass, kg
13	142	4	21	887.3
25	217	4	19	869.1
32	293	5	22	661.7
50	460	5	25	658.2
74	777	5	28	648.6
146	1708	5	26	513.0

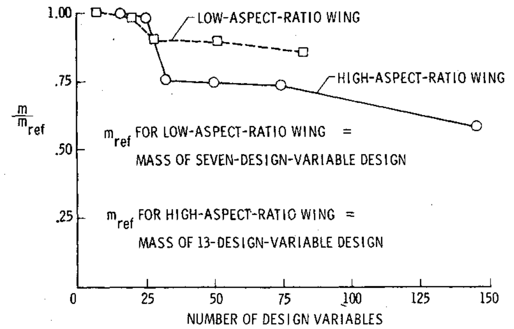

Fig. 6 Total mass of final designs as a function of design variables.

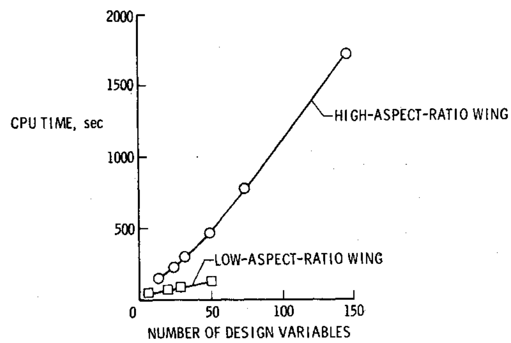

Fig. 7 Computational time as a function of number of design variables.

analyses. This case was also studied using finite-difference derivatives and no approximate methods and required 46 sec of CDC 6600 CPU time and 131 analyses. The final designs in each case were the same. Even though the number of analyses goes down by a factor of 10, the computer time improves only by a factor of 2 because each of the analyses with analytical derivatives includes a derivative calculation while the analyses with finite-difference derivatives did not. As the number of degrees of freedom increase the use of analytical derivatives will result in a greater savings in computer time when compared to a finite-difference approach.

Conclusions

In order to use mathematical programming methods successfully for large order structural design problems with a large number of design variables, it is necessary that efficient search techniques, approximate analysis methods, and a mechanism to integrate these component procedures be used. This paper presents a method that is an example of how these component procedures can be integrated.

A quadratic extended interior penalty function is presented that is especially well-suited for use with an efficient second-order resizing algorithm (Newton's method). The extended penalty function provides a mechanism that allows an interior penalty function formulation to recover from constraint violations that occur during the optimization process. It also permits the use of an initial design that violates the constraints.

Analytical derivative expressions for the first derivatives of the constraint functions used by the resizing algorithm are discussed and a first-order approximate analysis procedure for the one-dimensional search is also presented. Analytical derivatives and approximate analysis techniques make it possible to reduce significantly the total number of analyses required by the mathematical programming procedure.

The results given in the paper indicate that the integration of the preceding procedures not only leads to faster design solutions, but also allows significantly larger design problems to be solved than with earlier mathematical programming procedures. The progress that has been made in recent years toward reducing the computational times of the total optimization procedure indicates that mathematical programming techniques are becoming useful as a resizing tool for multiple constraint structural optimization problems with a large number of design variables.

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